Gridless Parameter Estimation in Partly Calibrated Rectangular Arrays

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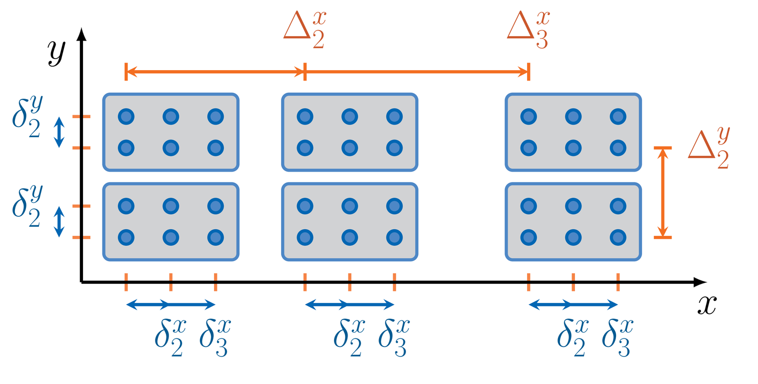
Introduction

- Proposal of a gridless sparse formulation for direction-of-arrival (DOA) estimation in partly calibrated rectangular arrays based on shift invariances
- Development of an efficient algorithm in the alternating direction method of multipliers (ADMM) framework

Mathematical Model and Notations

- Partly calibrated rectangular array (PCRA) with fully calibrated identical subarrays
  - $M = M_x \times M_y$: Total number of sensors
  - Δ_p^x (Δ_p^y) : *Unknown* intersubarray displacement between the pth and the first subarrays in x-axis (y-axis)
  - δ_l^x (δ_l^y) : *Known* intrasubarray displacement between the lth and the first sensors in x-axis (y-axis)



$M_x \times M_y$ PCRA composed of $P_x \times P_y$ subarrays of $L_x \times L_y$ sensors

- Distinct Directions-of-Arrival (DOAs) from N_S far-field narrowband sources with azimuth angle $\phi_i \in [-180°, 180°)$ and elevation angle $\theta_i \in [0°, 90°]$ for $i = 1, \ldots, N_\mathsf{S}$.
- Equivalent expression of DOA (ϕ_i, θ_i) in spatial frequencies (μ_i^x, μ_i^y) with

$$\mu_i^x = \pi \cos(\phi_i)\sin(\theta_i) \in [-\pi, \pi) \quad \text{and} \quad \mu_i^y = \pi \sin(\phi_i)\sin(\theta_i) \in [-\pi, \pi)$$

- Signal Model

$$\boldsymbol{Y} = \boldsymbol{A}(\boldsymbol{\mu})\boldsymbol{\Psi} + \boldsymbol{N}$$

$$\boldsymbol{\mu} = [\mu_1^x, \ldots, \mu_{N_\mathsf{S}}^x, \mu_1^y, \ldots, \mu_{N_\mathsf{S}}^y]^\mathsf{T}$$

$\boldsymbol{Y} \in \mathbb{C}^{M\times N}$: Received signal matrix
$\boldsymbol{\Psi} \in \mathbb{C}^{N_\mathsf{S}\times N}$: Source signal matrix
$\boldsymbol{N} \in \mathbb{C}^{M\times N}$: Sensor noise matrix
N : Number of available snapshots

- Steering matrix $\boldsymbol{A}(\boldsymbol{\mu}) = [\boldsymbol{a}(\mu_1^x, \mu_1^y), \ldots, \boldsymbol{a}(\mu_{N_\mathsf{S}}^x, \mu_{N_\mathsf{S}}^y)] \in \mathbb{C}^{M\times N_\mathsf{S}}$ with

$$\boldsymbol{a}(\mu_i^x, \mu_i^y) = \boldsymbol{a}_x(\mu_i^x) \otimes \boldsymbol{a}_y(\mu_i^y)$$

$$\boldsymbol{a}_x(\mu_i^x) = [1, \ldots, e^{j\mu_i^x\delta_{L_x}^x}, e^{j\mu_i^x\Delta_2^x}, \ldots, e^{j\mu_i^x(\Delta_{P_x}^x+\delta_{L_x}^x)}]^\mathsf{T} \in \mathbb{C}^{M_x}$$
$$\boldsymbol{a}_y(\mu_i^y) = [1, \ldots, e^{j\mu_i^y\delta_{L_y}^y}, e^{j\mu_i^y\Delta_2^y}, \ldots, e^{j\mu_i^y(\Delta_{P_y}^y+\delta_{L_y}^y)}]^\mathsf{T} \in \mathbb{C}^{M_y}$$

Shift Invariances in the PCRA

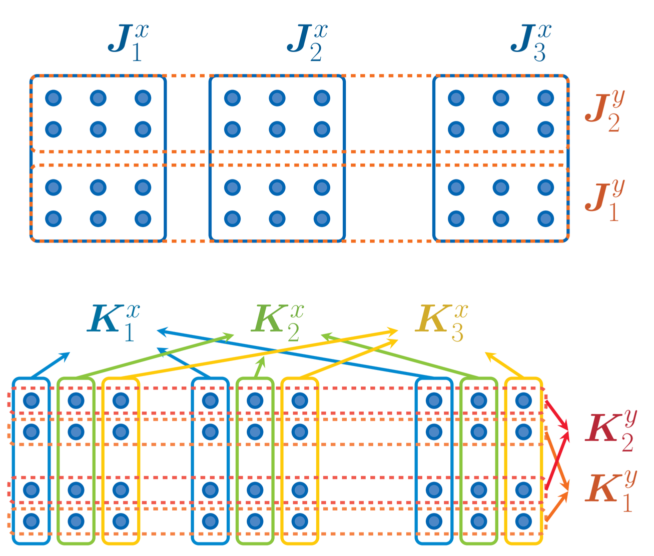
Shift subarrays:

$$(\boldsymbol{J}_p^x)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu}) = (\boldsymbol{J}_1^x)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu})\boldsymbol{\Phi}(\Delta_p^x\boldsymbol{\mu}^x), \quad p=2,\ldots,P_x$$
$$(\boldsymbol{J}_p^y)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu}) = (\boldsymbol{J}_1^y)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu})\boldsymbol{\Phi}(\Delta_p^y\boldsymbol{\mu}^y), \quad p=2,\ldots,P_y$$

Shift sensors within a subarray:

$$(\boldsymbol{K}_l^x)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu}) = (\boldsymbol{K}_1^x)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu})\boldsymbol{\Phi}(\delta_l^x\boldsymbol{\mu}^x), \quad l=2,\ldots,L_x$$
$$(\boldsymbol{K}_l^y)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu}) = (\boldsymbol{K}_1^y)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu})\boldsymbol{\Phi}(\delta_l^y\boldsymbol{\mu}^y), \quad l=2,\ldots,L_y$$

- $\boldsymbol{\mu}^x = [\mu_1^x, \ldots, \mu_{N_\mathsf{S}}^x]^\mathsf{T}$ and $\boldsymbol{\mu}^y = [\mu_1^y, \ldots, \mu_{N_\mathsf{S}}^y]^\mathsf{T}$
- $\boldsymbol{\Phi}(\boldsymbol{x}) = \mathrm{Diag}(e^{jx_1}, \ldots, e^{jx_N}) \in \mathbb{C}^{N\times N}$ for $\boldsymbol{x} \in \mathbb{R}^N$



Conventional Approach
ESPRIT-like methods performed on the sample covariance matrix $\widehat{\boldsymbol{R}} = \boldsymbol{Y}\boldsymbol{Y}^\mathsf{H}/N$ to recover the DOAs based on the shift invariances involving the known intrasubarray displacements δ_l^x and δ_l^y [HN98]

Grid-Based Sparse Formulation for Fully Calibrated Arrays

- Sample the field-of-view (FOV) in $K \gg N_\mathsf{S}$ directions with spatial frequencies $\boldsymbol{\nu} = [\nu_1^x, \ldots, \nu_K^x, \nu_1^y, \ldots, \nu_K^y]^\mathsf{T}$
- On-grid assumption: $\{(\mu_i^x, \mu_i^y)\}_{i=1}^{N_\mathsf{S}} \subset \{(\nu_k^x, \nu_k^y)\}_{k=1}^{K}$
- Sparse signal model

$$\boldsymbol{Y} = \boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{X} + \boldsymbol{N}$$

$\boldsymbol{X} \in \mathbb{C}^{K\times N}$: Row-sparse representation of $\boldsymbol{\Psi}$
$\boldsymbol{A}(\boldsymbol{\nu}) \in \mathbb{C}^{M\times K}$: Steering matrix for sampled directions $\boldsymbol{\nu}$

- $\ell_{2,1}$-mixed-norm minimization

$$\widehat{\boldsymbol{X}} = \underset{\boldsymbol{X}\in\mathbb{C}^{K\times N}}{\mathrm{argmin}} \ \tfrac{1}{2}\|\boldsymbol{Y} - \boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{X}\|_\mathsf{F}^2 + \lambda\sqrt{N}\|\boldsymbol{X}\|_{2,1}$$

$\|\boldsymbol{X}\|_{2,1} = \sum_{k=1}^K \|\boldsymbol{x}_k\|_2$ for $\boldsymbol{X} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K]^\mathsf{T}$
$\lambda > 0$: Regularization parameter

- SPARROW reformulation [SPP18]

$$\widehat{\boldsymbol{S}} = \underset{\boldsymbol{S}\in\mathbb{D}_+^K}{\mathrm{argmin}} \ \mathrm{tr}\left((\boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{S}\boldsymbol{A}(\boldsymbol{\nu})^\mathsf{H} + \lambda\boldsymbol{I}_M)^{-1}\widehat{\boldsymbol{R}}\right) + \mathrm{tr}(\boldsymbol{S})$$

$\mathbb{D}_+^K$: Set of $K \times K$ nonnegative diagonal matrices
$\widehat{\boldsymbol{S}} = \frac{1}{\sqrt{N}}\mathrm{Diag}(\|\widehat{\boldsymbol{x}}_1\|_2, \ldots, \|\widehat{\boldsymbol{x}}_K\|_2)$

Shift-Invariant SPARROW (SI-SPARROW)

- Gridless relaxation of SPARROW $\Longrightarrow$ Shift-Invariant SPARROW (SI-SPARROW)

$$\min_{\boldsymbol{S}\in\mathbb{D}_+^N, \boldsymbol{A}\in\mathcal{A}^K, \boldsymbol{Q}\in\mathbb{S}_+^M} M\,\mathrm{tr}\left((\boldsymbol{Q} + \lambda\boldsymbol{I}_M)^{-1}\widehat{\boldsymbol{R}}\right) + \mathrm{tr}(\boldsymbol{Q})$$

subject to $\boldsymbol{Q} = \boldsymbol{A}\boldsymbol{S}\boldsymbol{A}^\mathsf{H}$ $\xrightarrow{\text{Relaxation}}$ $\boldsymbol{Q} \in \mathcal{T}^M$

The shfit-invariant subspace $\mathcal{T}^M$ is the set of $\boldsymbol{Q} \in \mathbb{S}^M$ that satisfies

$$(\boldsymbol{J}_p^x)^\mathsf{T}\boldsymbol{Q}\boldsymbol{J}_p^x = (\boldsymbol{J}_1^x)^\mathsf{T}\boldsymbol{Q}\boldsymbol{J}_1^x, \quad p = 2, \ldots, P_x$$
$$(\boldsymbol{J}_p^y)^\mathsf{T}\boldsymbol{Q}\boldsymbol{J}_p^y = (\boldsymbol{J}_1^y)^\mathsf{T}\boldsymbol{Q}\boldsymbol{J}_1^y, \quad p = 2, \ldots, P_y$$
$$(\boldsymbol{K}_l^x)^\mathsf{T}\boldsymbol{Q}\boldsymbol{K}_l^x = (\boldsymbol{K}_1^x)^\mathsf{T}\boldsymbol{Q}\boldsymbol{K}_1^x, \quad l = 2, \ldots, L_x$$
$$(\boldsymbol{K}_l^y)^\mathsf{T}\boldsymbol{Q}\boldsymbol{K}_l^y = (\boldsymbol{K}_1^y)^\mathsf{T}\boldsymbol{Q}\boldsymbol{K}_1^y, \quad l = 2, \ldots, L_y$$
$$q_{ii} = q_{11}, \quad i = 2, \ldots, M$$

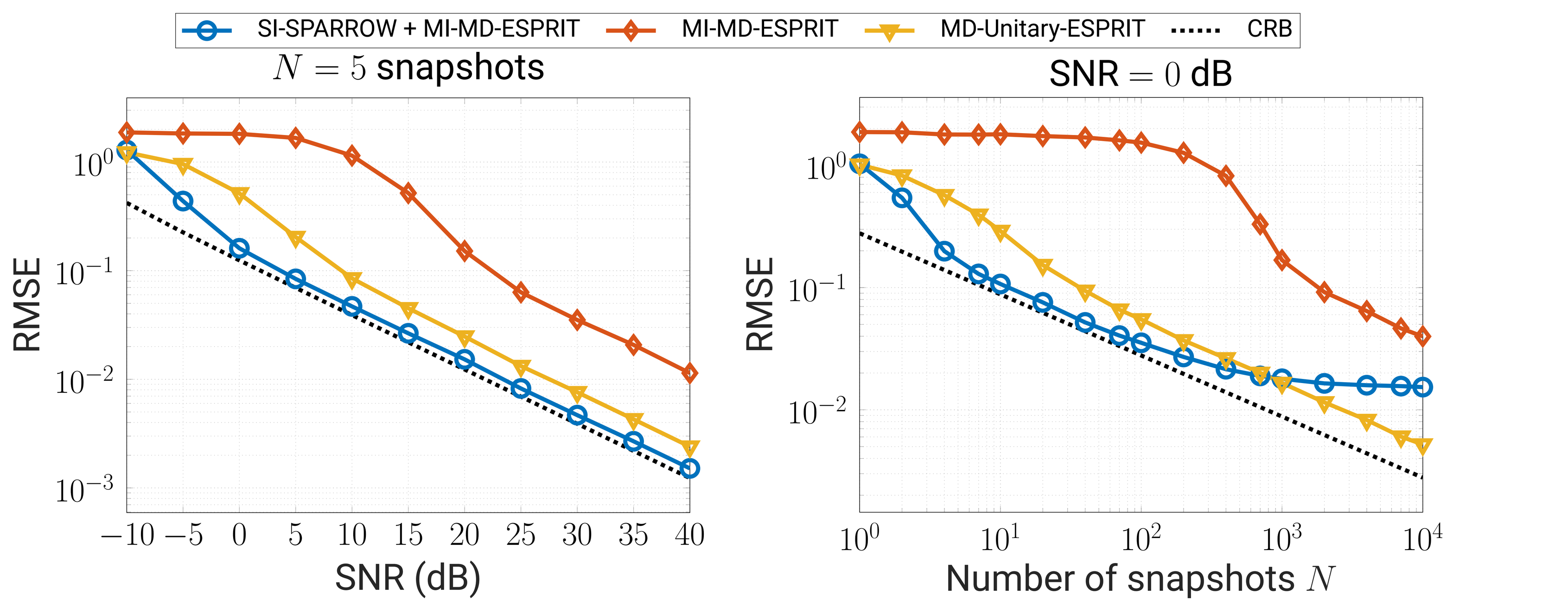
$\mathcal{A}^K = \{\boldsymbol{A}(\boldsymbol{\nu}) \mid \boldsymbol{\nu} \in [-\pi, \pi]^{2K}, (\nu_i^x, \nu_i^y) \neq (\nu_j^x, \nu_j^y)\ \forall i, j = 1, \ldots, K,\ i \neq j\}$: Array manifold with K distinct DOAs

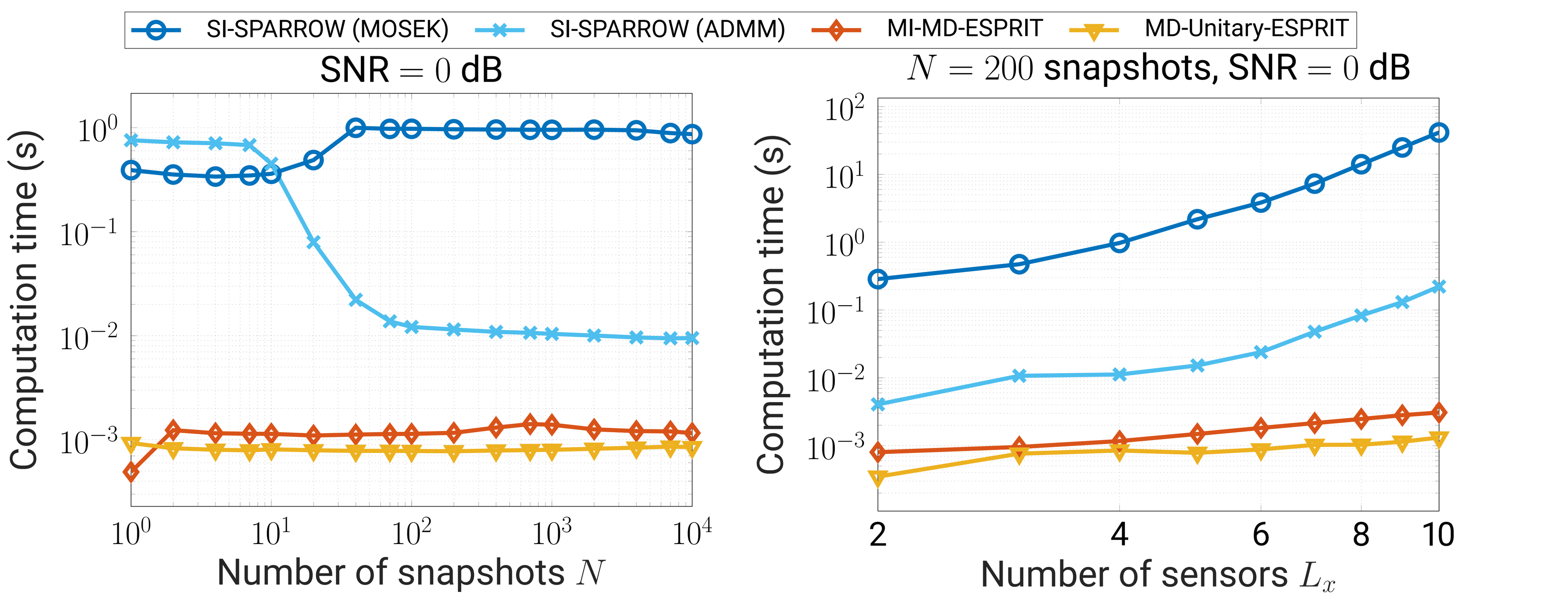
- ESPRIT-like methods performed on $\boldsymbol{Q}$ to recover DOAs
- Solution approaches for SI-SPARROW:

Semidefinite Programming (SDP)	Alternating Direction Method of Multipliers (ADMM)
$\min_{\boldsymbol{Q}\in\mathbb{S}_+^M\cap\mathcal{T}^M, \boldsymbol{T}\in\mathbb{S}_+^N} \frac{M}{N}\mathrm{tr}(\boldsymbol{T}) + \mathrm{tr}(\boldsymbol{Q})$ s.t. $\begin{bmatrix}\boldsymbol{T} & \boldsymbol{Y}^\mathsf{H} \\ \boldsymbol{Y} & \boldsymbol{Q} + \lambda\boldsymbol{I}_M\end{bmatrix} \succeq 0$	$\min_{\boldsymbol{Q}\in\mathcal{T}^M, \boldsymbol{Z}\in\mathbb{S}^M} M\,\mathrm{tr}\left((\boldsymbol{Q} + \lambda\boldsymbol{I}_M)^{-1}\widehat{\boldsymbol{R}}\right) + \mathrm{tr}(\boldsymbol{Q}) + \mathbb{I}_{\mathbb{S}_+^M}(\boldsymbol{Z})$ s.t. $\boldsymbol{Q} - \boldsymbol{Z} = \boldsymbol{0}$

Simulation Results

- SDP problems solved by MOSEK solver
- PCRA composed of 2×2 uniform rectangular subarrays of 4×2 sensors
- Correlated sources with correlation coefficient 0.99
- Comparison methods: Multi-Invariance Multidimensional ESPRIT (MI-MD-ESPRIT) and Multidimensional Unitary ESPRIT (MD-Unitary-ESPRIT) performed on the sample covariance matrix





References

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[SPP18] Christian Steffens, Marius Pesavento, and Marc E. Pfetsch. A compact formulation for the $\ell_{2,1}$ mixed-norm minimization problem. *IEEE Trans. Signal Process.*, 66(6):1483–1497, March 2018.

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