Blind Phase-Offset Estimation in Sparse Partly Calibrated Arrays

Tianyi Liu and Marius Pesavento

TECHNISCHE UNIVERSITÄT DARMSTADT

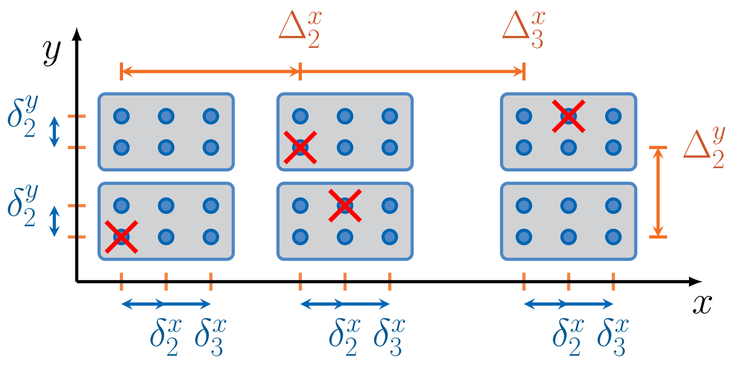
Communication Systems Group, Technical University of Darmstadt, Darmstadt, Germany

Introduction

- Joint direction-of-arrival (DOA) and subarray phase-offset estimation in sparse partly calibrated arrays
- Proposal of a deterministic maximum likelihood (DML) estimator via semidefinite relaxation for subarray phase-offset estimation

Mathematical Model and Notations

- Partly calibrated rectangular array (PCRA) with fully calibrated identical subarrays
  - $M = M_x \times M_y$: Total number of sensors
  - Δ_p^x (Δ_p^y) : *Known* intersubarray displacement between the pth and the first subarrays in x-axis (y-axis)
  - δ_l^x (δ_l^y) : *Known* intrasubarray displacement between the lth and the first sensors in x-axis (y-axis)
  - τ_p^x: *Unknown* phase-offset between the pth and the first subarrays in x-axis
  - Only a subset $\mathcal{M}' = \{i_1, \ldots, i_{M'}\}$ of $M' \leq M$ sensors is observable



$M_x \times M_y$ PCRA composed of $P_x \times P_y$ subarrays of $L_x \times L_y$ sensors

- Distinct Directions-of-Arrival (DOAs) from N_S far-field narrowband sources with azimuth angle $\phi_i \in [-180°, 180°)$ and elevation angle $\theta_i \in [0°, 90°]$ for $i = 1, \ldots, N_\text{S}$.
- Equivalent expression of DOA (ϕ_i, θ_i) in spatial frequencies (μ_i^x, μ_i^y) with

$$\mu_i^x = \pi \cos(\phi_i)\sin(\theta_i) \in [-\pi, \pi) \quad \text{and} \quad \mu_i^y = \pi \sin(\phi_i)\sin(\theta_i) \in [-\pi, \pi)$$

- Signal Model

$$\boldsymbol{Y} = \boldsymbol{J}_{\mathcal{M}'}^\mathsf{T} \bar{\boldsymbol{A}}(\boldsymbol{\mu}, \boldsymbol{\tau}^x)\boldsymbol{\Psi} + \boldsymbol{N}$$

$$\boldsymbol{\mu} = [\mu_1^x, \ldots, \mu_{N_\text{S}}^x, \mu_1^y, \ldots, \mu_{N_\text{S}}^y]^\mathsf{T}$$
$$\boldsymbol{\tau}^x = [1, \tau_2^x, \ldots, \tau_{P_x}^x]^\mathsf{T}$$

$\boldsymbol{J}_{\mathcal{M}'} = [\boldsymbol{e}_{M,i_1}, \ldots, \boldsymbol{e}_{M,i_{M'}}] \in \mathbb{R}^{M \times M'}$
$\boldsymbol{Y} \in \mathbb{C}^{M' \times N}$: Received signal matrix
$\boldsymbol{\Psi} \in \mathbb{C}^{N_\text{S} \times N}$: Source signal matrix
$\boldsymbol{N} \in \mathbb{C}^{M' \times N}$: Sensor noise matrix
N : Number of available snapshots

- Steering matrix $\bar{\boldsymbol{A}}(\boldsymbol{\mu}, \boldsymbol{\tau}^x) = \boldsymbol{\Phi}(\bar{\boldsymbol{\tau}}^x)\boldsymbol{A}(\boldsymbol{\mu}) \in \mathbb{C}^{M \times N_\text{S}}$:

  $\boldsymbol{A}(\boldsymbol{\mu}) = [\boldsymbol{a}(\mu_1^x, \mu_1^y), \ldots, \boldsymbol{a}(\mu_{N_\text{S}}^x, \mu_{N_\text{S}}^y)] \in \mathbb{C}^{M \times N_\text{S}}$: Fully calibrated steering matrix

  $\boldsymbol{\Phi}(\bar{\boldsymbol{\tau}}^x) \in \mathbb{C}^{M \times M}$: Diagonal matrix containing unknown phase-offsets

  ⋆ $\bar{\boldsymbol{\tau}}^x = \boldsymbol{T}\boldsymbol{\tau}^x \in \mathbb{R}^{M \times 1}$, $\boldsymbol{T}$ maps the unknown phase-offsets to each sensor

  ⋆ $\boldsymbol{\Phi}(\boldsymbol{x}) = \text{Diag}(\text{e}^{\text{j}x_1}, \ldots, \text{e}^{\text{j}x_N}) \in \mathbb{C}^{N \times N}$ for $\boldsymbol{x} \in \mathbb{R}^N$

Shift Invariances in the PCRA

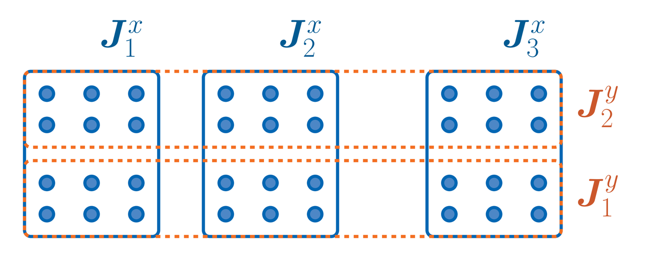
Shift subarrays:

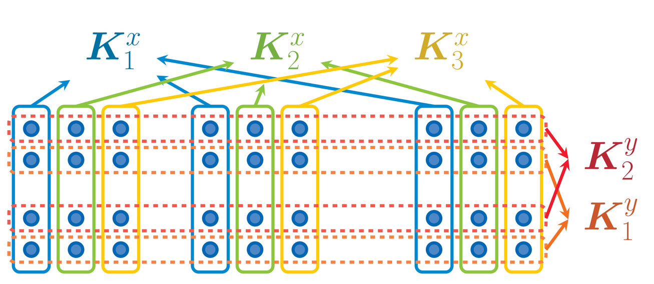
$$(\boldsymbol{J}_p^x)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu}) = (\boldsymbol{J}_1^x)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu})\boldsymbol{\Phi}(\Delta_p^x \boldsymbol{\mu}^x)\text{e}^{\text{j}\tau_p^x}, \ p=2, \ldots, P_x$$
$$(\boldsymbol{J}_p^y)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu}) = (\boldsymbol{J}_1^y)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu})\boldsymbol{\Phi}(\Delta_p^y \boldsymbol{\mu}^y), \quad p=2, \ldots, P_y$$

Shift sensors within a subarray:

$$(\boldsymbol{K}_l^x)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu}) = (\boldsymbol{K}_1^x)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu})\boldsymbol{\Phi}(\delta_l^x \boldsymbol{\mu}^x), \ l=2, \ldots, L_x$$
$$(\boldsymbol{K}_l^y)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu}) = (\boldsymbol{K}_1^y)^\mathsf{T}\boldsymbol{A}(\boldsymbol{\mu})\boldsymbol{\Phi}(\delta_l^y \boldsymbol{\mu}^y), \ l=2, \ldots, L_y$$

- $\boldsymbol{\mu}^x = [\mu_1^x, \ldots, \mu_{N_\text{S}}^x]^\mathsf{T}$ and $\boldsymbol{\mu}^y = [\mu_1^y, \ldots, \mu_{N_\text{S}}^y]^\mathsf{T}$





Conventional Approach

ESPRIT-like methods performed on the sample covariance matrix $\widehat{\boldsymbol{R}} = \boldsymbol{Y}\boldsymbol{Y}^\mathsf{H}/N$ to recover the DOAs based on the shift invariances involving the known intrasubarray displacements δ_l^x and δ_l^y [HN98]

Grid-Based Sparse Formulation for Fully Calibrated Arrays

- Sample the field-of-view (FOV) in $K \gg N_\text{S}$ directions with spatial frequencies $\boldsymbol{\nu} = [\nu_1^x, \ldots, \nu_K^x, \nu_1^y, \ldots, \nu_K^y]^\mathsf{T}$
- On-grid assumption: $\{(\mu_i^x, \mu_i^y)\}_{i=1}^{N_\text{S}} \subset \{(\nu_k^x, \nu_k^y)\}_{k=1}^{K}$
- Sparse signal model

$$\boldsymbol{Y} = \boldsymbol{J}_{\mathcal{M}'}^\mathsf{T}\boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{X} + \boldsymbol{N}$$

$\boldsymbol{X} \in \mathbb{C}^{K \times N}$: Row-sparse representation of $\boldsymbol{\Psi}$
$\boldsymbol{A}(\boldsymbol{\nu}) \in \mathbb{C}^{M \times K}$: Steering matrix for sampled directions $\boldsymbol{\nu}$

- $\ell_{2,1}$-mixed-norm minimization $\Longrightarrow$ SPARROW reformulation [SPP18]

$$\min_{\boldsymbol{X} \in \mathbb{C}^{K \times N}} \tfrac{1}{2}\|\boldsymbol{Y} - \boldsymbol{J}_{\mathcal{M}'}^\mathsf{T}\boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{X}\|_\text{F}^2 + \lambda\sqrt{N}\|\boldsymbol{X}\|_{2,1}$$

$$\Longrightarrow \min_{\boldsymbol{S} \in \mathbb{D}_+^K} \text{tr}\left((\boldsymbol{J}_{\mathcal{M}'}^\mathsf{T}\boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{S}\boldsymbol{A}(\boldsymbol{\nu})^\mathsf{H}\boldsymbol{J}_{\mathcal{M}'} + \lambda\boldsymbol{I})^{-1}\widehat{\boldsymbol{R}}\right) + \text{tr}(\boldsymbol{S})$$

$\mathbb{D}_+^K$: Set of $K \times K$ nonnegative diagonal matrices

Relation between the optimal solutions: $\widehat{\boldsymbol{S}} = \frac{1}{\sqrt{N}}\text{Diag}(\|\widehat{\boldsymbol{x}}_1\|_2, \ldots, \|\widehat{\boldsymbol{x}}_K\|_2)$

Shift-Invariant SPARROW (SI-SPARROW) for DOA Estimation

- Gridless relaxation of SPARROW $\Longrightarrow$ Shift-Invariant SPARROW (SI-SPARROW)

$$\min_{\boldsymbol{S} \in \mathbb{D}_+^N, \boldsymbol{A} \in \mathcal{A}^K, \boldsymbol{Q} \in \mathbb{S}_+^M, \boldsymbol{\tau}^x} M \,\text{tr}\left((\boldsymbol{J}_{\mathcal{M}'}^\mathsf{T}\boldsymbol{Q}\boldsymbol{J}_{\mathcal{M}'} + \lambda\boldsymbol{I}_{M'})^{-1}\widehat{\boldsymbol{R}}\right) + \text{tr}(\boldsymbol{Q})$$

subject to $\boldsymbol{Q} = \boldsymbol{\Phi}(\bar{\boldsymbol{\tau}}^x)\boldsymbol{A}\boldsymbol{S}\boldsymbol{A}^\mathsf{H}\boldsymbol{\Phi}(\bar{\boldsymbol{\tau}}^x)^\mathsf{H}$

↓ Relaxation

$\boldsymbol{Q} \in \mathcal{T}^M$

The shfit-invariant subspace $\mathcal{T}^M$ is the set of $\boldsymbol{Q} \in \mathbb{S}^M$ that satisfies

$$\begin{aligned}
(\boldsymbol{J}_p^x)^\mathsf{T}\boldsymbol{Q}\boldsymbol{J}_p^x &= (\boldsymbol{J}_1^x)^\mathsf{T}\boldsymbol{Q}\boldsymbol{J}_1^x, & p &= 2, \ldots, P_x \\
(\boldsymbol{J}_p^y)^\mathsf{T}\boldsymbol{Q}\boldsymbol{J}_p^y &= (\boldsymbol{J}_1^y)^\mathsf{T}\boldsymbol{Q}\boldsymbol{J}_1^y, & p &= 2, \ldots, P_y \\
(\boldsymbol{K}_l^x)^\mathsf{T}\boldsymbol{Q}\boldsymbol{K}_l^x &= (\boldsymbol{K}_1^x)^\mathsf{T}\boldsymbol{Q}\boldsymbol{K}_1^x, & l &= 2, \ldots, L_x \\
(\boldsymbol{K}_l^y)^\mathsf{T}\boldsymbol{Q}\boldsymbol{K}_l^y &= (\boldsymbol{K}_1^y)^\mathsf{T}\boldsymbol{Q}\boldsymbol{K}_1^y, & l &= 2, \ldots, L_y \\
q_{ii} &= q_{11}, & i &= 2, \ldots, M
\end{aligned}$$

$\mathcal{A}^K = \{\boldsymbol{A}(\boldsymbol{\nu}) \mid \boldsymbol{\nu} \in [-\pi, \pi]^{2K}, (\nu_i^x, \nu_i^y) \neq (\nu_j^x, \nu_j^y)\ \forall i, j = 1, \ldots, K,\ i \neq j\}$: Array manifold with K distinct DOAs

- Precise calibration information not involved in SI-SPARROW
- Implicit sensor completion
- Solution approaches for SI-SPARROW: Semidefinite Programming [SPP18]; ADMM [LDA+24]
- ESPRIT-like methods performed on the solution $\widehat{\boldsymbol{Q}}$ to recover DOAs $\widehat{\boldsymbol{\mu}}$

DML for Subarray Timing-Offset Calibration

- Deterministic Maximum Likelihood (DML) estimation using $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{Q}}$:

$$\min_{\boldsymbol{\tau}^x} \ \text{tr}\left(\widehat{\boldsymbol{Q}}\mathcal{P}_{\bar{\boldsymbol{A}}}^{\perp}(\widehat{\boldsymbol{\mu}}, \boldsymbol{\tau}^x)\right)$$

↓ Semidefinite Relaxation

$$\min_{\boldsymbol{Z} \in \mathbb{S}_+^{P_x}} \ \text{tr}\left(\widehat{\boldsymbol{Q}}\left(\mathcal{P}_{\boldsymbol{A}}^{\perp}(\widehat{\boldsymbol{\mu}}) \odot (\boldsymbol{T}\boldsymbol{Z}\boldsymbol{T}^\mathsf{T})\right)\right) \quad \text{s.t.} \ z_{i,i} = 1, i = 1, \ldots, P_x$$

Projector onto the orthogonal complement of range space of $\bar{\boldsymbol{A}}(\widehat{\boldsymbol{\mu}}, \boldsymbol{\tau}^x)$:

$$\begin{aligned}
\mathcal{P}_{\bar{\boldsymbol{A}}}^{\perp}(\widehat{\boldsymbol{\mu}}, \boldsymbol{\tau}^x) &= \boldsymbol{\Phi}(\bar{\boldsymbol{\tau}}^x)\mathcal{P}_{\boldsymbol{A}}^{\perp}(\widehat{\boldsymbol{\mu}})\boldsymbol{\Phi}^\mathsf{H}(\bar{\boldsymbol{\tau}}^x) \\
&= \mathcal{P}_{\boldsymbol{A}}^{\perp}(\widehat{\boldsymbol{\mu}}) \odot \left(\text{e}^{\text{j}\bar{\boldsymbol{\tau}}^x}(\text{e}^{\text{j}\bar{\boldsymbol{\tau}}^x})^\mathsf{H}\right) \\
&= \mathcal{P}_{\boldsymbol{A}}^{\perp}(\widehat{\boldsymbol{\mu}}) \odot \left(\boldsymbol{T}\boldsymbol{Z}(\boldsymbol{\tau}^x)\boldsymbol{T}^\mathsf{T}\right)
\end{aligned}$$

with $\boldsymbol{Z}(\boldsymbol{\tau}^x) = \text{e}^{\text{j}\boldsymbol{\tau}^x}\left(\text{e}^{\text{j}\boldsymbol{\tau}^x}\right)^\mathsf{H}$

- DML estimation using $\widehat{\boldsymbol{\mu}}$ and the sample covariance matrix $\widehat{\boldsymbol{R}}$ with unobservable sensors:

$$\min_{\boldsymbol{Z} \in \mathbb{S}_+^{P_x}} \ \text{tr}\left(\widehat{\boldsymbol{R}}\left(\mathcal{P}_{\boldsymbol{A}'}^{\perp}(\widehat{\boldsymbol{\mu}}) \odot (\boldsymbol{T}'\boldsymbol{Z}\boldsymbol{T}'^\mathsf{T})\right)\right) \quad \text{s.t.} \ z_{i,i} = 1, i = 1, \ldots, P_x$$

$\boldsymbol{A}'(\widehat{\boldsymbol{\mu}}) = \boldsymbol{J}_{\mathcal{M}'}^\mathsf{T}\boldsymbol{A}(\widehat{\boldsymbol{\mu}})$ and $\boldsymbol{T}' = \boldsymbol{J}_{\mathcal{M}'}^\mathsf{T}\boldsymbol{T}$

Simulation Results

- PCRA composed of 3×2 uniform rectangular subarrays of 4×2 sensors
- 2 correlated sources with correlation coefficient 0.99
- Comparison method: Multi-Invariance Multidimensional ESPRIT (MI-MD-ESPRIT) performed on the sample covariance matrix

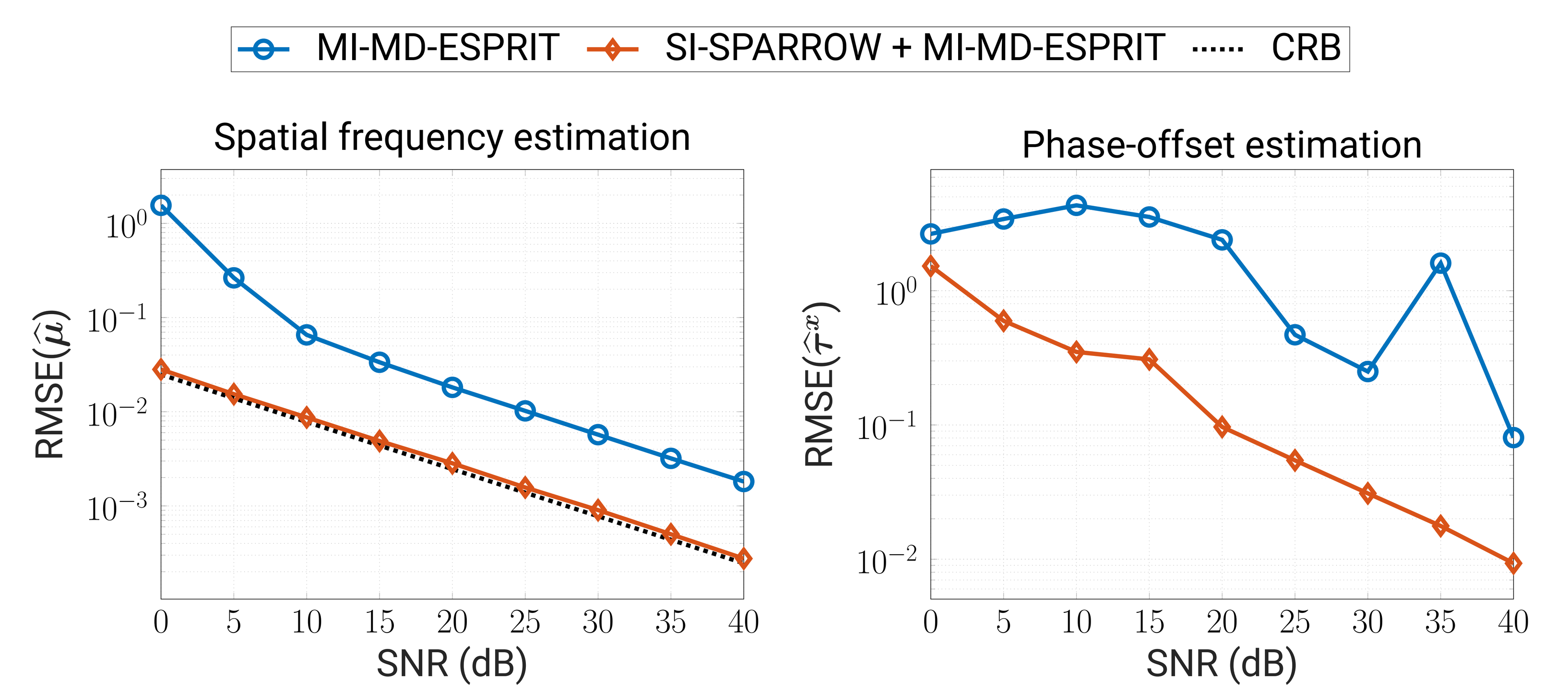


Fig. 1: Error performance w.r.t. SNR for $N = 100$ snapshots.

References

[HN98] M. Haardt and J. A. Nossek. Simultaneous Schur decomposition of several nonsymmetric matrices to achieve automatic pairing in multidimensional harmonic retrieval problems. *IEEE Trans. Signal Process.*, 46(1):161–169, 1998.

[LDA+24] Tianyi Liu, Sai Pavan Deram, Khaled Ardah, Martin Haardt, Marc E. Pfetsch, and Marius Pesavento. Gridless Parameter Estimation in Partly Calibrated Rectangular Arrays, 2024.

[SPP18] Christian Steffens, Marius Pesavento, and Marc E. Pfetsch. A compact formulation for the $\ell_{2,1}$ mixed-norm minimization problem. *IEEE Trans. Signal Process.*, 66(6):1483–1497, March 2018.

This work was supported in part by the DFG PRIDE Project PE 2080/2-1 and in part by the EXPRESS II Project within the DFG priority program CoSIP (DFG-SPP 1798).
2024 13th IEEE Sensor Array and Multichannel Signal Processing (SAM)